SUFFICIENT CONDITIONS FOR THE OSCILLATION OF SOME NONLINEAR DIFFERENCE EQUATIONS

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Abstract: *Our purpose in this paper is to give the sufficient conditions for the oscillation of solutions of nonlinear difference equation of the form:*

$$\Delta(a_n \Delta u_n) + b_n f(u_{n-\tau_n}) = 0 \text{ , } n = 0, 1, 2,\ldots \tag{01}$$

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1. INTRODUCTION

In this note we consider the nonlinear difference equation of the form (01).
Where: Δ denotes the forward difference operator, $\Delta u_n = u_{n+1} - u_n$ for any sequence (u_n) of real numbers;
(b_n) is a sequence of real numbers;
(τ_n) - is a sequence of integers such that:
$\lim_{n\to\infty}(n-\tau_n) = \infty$;
(a_n) - is a sequence of positive numbers and
$R_n = \sum_{k=o}^{n-1} \frac{1}{a_k} \to \infty$, as n → ∞;
$f:R \to R$ - is a continuous with
$uf(u) > 0$ $(u \neq 0)$.

By a solution of Equation (01) we mean a sequence (u_n) which is defined for
$n \geq \min_{i\geq 0}(i-\tau_i)$
and satisfies Equation (1) for all large n.

A nontrivial solution (u_n) of (1) is said to be oscillatory if for every $n_0 > 0$ there exists an $n > n_0$ such $u_n\, u_{n+1} \leq 0$. Otherwise it is called nonoscillatory.

In several recent papers the oscillatory behaviour of solutions of nonlinear difference quations have been discussed e.g. see [2,3,5,6]. Our purpose in this paper is to give the sufficient conditions for the oscillation of solutions of Equation (01).

2. MAIN RESULTS

Theorem 1**.** Assume that

i. $b_n \geq 0$ and $\sum_{n=0}^{\infty} b_n = \infty$,

ii. $\liminf_{|u|\to\infty} |f(u)| > 0$**.**

Then every solution of Equation (01) is oscillatory.

Proof: Assume, that Equation (01) has nonoscillatory solution (u_n), and we assume that (u_n) is eventually positive. Then there is a positive integer n_0 such that

$$u_{n-\tau_n} > 0,\ \text{for } n > n_0 \tag{02}$$

From the Equation (01) we have:
$\Delta(a_n\Delta u_n) = -b_n f(u_{n-r_n})$, $n > n_0$ and so $(a_n\Delta u_n)$ is an eventually nonincreasing sequence. We first show that $a_n\Delta u_n \geq 0$ for $n \geq n_0$. In fact, if there is an $n_1 \geq n_0$ such that $a_n\Delta u_n = c < 0$ and $a_n\Delta u_n$ ≤ c for $n > n_1$ that is $\Delta u_n \leq \frac{c}{a_n}$ and hence

$$u_n \leq u_{n_1} + c\sum_{k=n_1}^{n-1}\frac{1}{a_k} \to -\infty \text{ as n}\to\infty$$

which contradicts the fact that $x_n > 0$ for $n > n_1$. Hence $a_{n_1}\Delta u_{n_1} \geq 0$ for $n \geq n_0$. Therefore we obtain:

$u_{n-\tau_n} > 0$, $\Delta u_n \geq 0$,

$\Delta(a_n \Delta u_n) \leq 0$ for $n \geq n_0$.

Let $L = \lim_{n\to\infty} u_n$.

Then $L > 0$ is finite or infinite.

Case 1: $L > 0$ is finite.

From the continuity of function f(u) we have: $\lim_{n\to\infty} f(u_{n-\tau_n}) = f(L) > 0$. Thus, we may choose a positive integer $n_3(\geq n_0)$ such that:

$$f(u_{n-\tau_n}) > \frac{1}{2} f(L), \; n \geq n_3 \quad (03)$$

By substituting (03) into Equation (01) we obtain:

$$\Delta(a_n \Delta u_n) + \frac{1}{2} f(L) b_n \leq 0, \; n \geq n_3 \quad (04)$$

Summing up both sides of (04) from n_3 to $n(\geq n_3)$, we obtain:

$$a_{n+1}\Delta u_{n+1} - a_{n_3}\Delta u_{n_3} + \frac{1}{2} f(L) \sum_{i=n_3}^{n} b_i \leq 0$$

and so

$$\frac{1}{2} f(L) \sum_{i=n_3}^{n} b_i \leq a_{n_3}\Delta u_{n_3}, \; n \geq n_3,$$

which contradicts (i).

Case 2: $L = \infty$.

For this case, from the condition (ii) we have $\liminf_{n\to\infty} f(u_{n-\tau_n}) > 0$ and so we may choose a positive constant c and a positive integer n_4 sufficiently large such that:

$$f(u_{n-\tau_n}) \geq c \text{ for } n \geq n_4. \quad (05)$$

Substituting (5) into Equation (1) we have:

$$\Delta(a_n \Delta u_n) + c b_n \leq 0, \; n \leq n_4.$$

Using the similar argument as that of Case 1 we may obtain a contradiction to the condition (i). This completes the proof.

Theorem 2. Assume, that

i. $b_n \geq 0$ and then every bounded solution of (01) is oscillatory.

Proof: Proceeding as in the proof of Theorem1 with assumption that (u_n) is a bounded nonoscillatory solution of (01) we get the inequality (04) and so we obtain:

$$R_n \Delta(a_n \Delta u_n) + \frac{1}{2} f(L) R_n b_n \leq 0, \; n \geq n_3. \quad (06)$$

It is easy to see that:

$$R_n \Delta(a_n \Delta u_n) \geq \Delta(R_n a_n \Delta u_n) - a_n \Delta u_n \Delta R_n. \quad (07)$$

From inequalities (6) and (7) we deduce:

$$\sum_{k=n_3}^{n} \Delta R_k a_k \Delta u_k) - \sum_{k=n_3}^{n} \Delta u_k + \frac{1}{2} f(L) \sum_{k=n_3}^{n} R_k b_k \leq 0,$$

$n \geq n_3$.

which implies

$$\frac{1}{2} f(L) \sum_{k=n_3}^{n} R_k b_k \leq u_{n+1} + R_{n_3} a_{n_3} \Delta u_{n_3}) - u_{n_3}, \; n \geq n_3.$$

Hence there exists a constant c such that:

$$\sum_{k=n_3}^{n} R_k b_k \leq c, \text{ for all } n \geq n_3,$$

contrary to the assumption of the theorem.

Theorem 3. Assume that

i. $(n - \tau_n)$ is nondecreasing, where $\tau_n \in \{0, 1, 2,\ldots\}$;

ii. there is a subsequence of (a_n), say (u_{n_k}) such that $u_{n_k} \leq 1$ for $k = 0, 1, 2,\ldots$;

iii. $\sum_{n=0}^{\infty} b_n = \infty$,

iv. f is nondecreasing and there is a nonnegative constant M such that:

$$\limsup_{u\to 0} \frac{u}{f(u)} = M \quad (08)$$

Then the difference (Δu_n) of every solution (u_n) of Equation (1) oscillates.

Proof: If not, then Equation (01) has a solution (u_n) such that its difference (Δu_n) is nonoscillatory. Assume first that the sequence (Δu_n) is eventually negative. Then there is a positive integer n_0 such that $\Delta u_n < 0$ $n \geq n_0$ and so (u_n) is decreasing for $n \geq n_0$ which implies that (u_n) is also nonoscillatory. Set

$$w_n = \frac{a_n}{f(x_{n-\tau_n})}, \; n \geq n_1 \geq n_0 \quad (09)$$

then

$$\Delta w_n = \frac{a_{n+1}\Delta u_n}{f(u_{n+1-\tau_{n+1}})} - \frac{a_n \Delta u_n}{f(u_{n-\tau_n})}$$

$$= \frac{\Delta(a_n \Delta u_n)}{f(u_{n-\tau_n})} + a_{n+1}\Delta u_{n+1}$$

$$\frac{f(u_{n-\tau_n})-f(u_{n+1-\tau_{n+1}})}{f(u_{n+1-\tau_{n+1}})f(u_{n-\tau_n})} \le \frac{\Delta(a_n\Delta u_n)}{f(u_{n-\tau_n})} = -b_n \quad (10)$$

$n \ge n_1$

Summing up both sides of (10) from n_1 to n, we have: $w_{n+1} - w_{n_1} \le -\sum_{i=n_1}^{n} b_i$ and, by (iii), we get

$$\lim_{x\to\infty} w_n = -\infty, \quad (11)$$

which implies that eventually

$$f(u_{n-\tau_n}) > 0 \quad (12)$$

and therefore $u_{n-\tau_n} > 0$. By (11), we can choose $n_2(\ge n_1)$ suchthat $w_n \le -(M+1)$, $n \ge n_2$. That is:

$$a_n\Delta u_n + (M+1)f(u_{n-\tau_n}) \le 0,\ n \ge n_2 \quad (13)$$

Set $\lim_{x\to\infty} u_n = L.$ Then L ≥ 0. Now we prove that L = 0. If L > 0, then we have

$$\lim_{n\to\infty} f(u_{n-\tau_n}) = f(L) > 0,$$

by the continuity of f(u). Choosing an n_3 sufficiently large, such that:

$$f(u_{n-\tau_n}) > \frac{1}{2} f(L),\ n \ge n_3 \quad (14)$$

and substituting (14) into (13), we have:

$$\Delta u_n + \frac{1}{2a_n}(M+1)f(L) \le 0,\ n \ge n_2 \quad (15)$$

Summing up both sides of (15) from n_3 to n we get

$$u_{n+1} - u_{n_3} + \frac{1}{2a_n}(M+1)f(L)\sum_{i=n_3}^{n}\frac{1}{a_i} \le 0$$

which implies that $\lim_{n\to\infty} u_n = -\infty$.

This contradicts (12). Hence $\lim_{n\to\infty} u_n = 0$.

By the assumptions we have:

$$\limsup_{n\to\infty} \frac{u_{n-\tau_n}}{f(u_{n-\tau_n})} \le M$$

From this we can choose n_4, such that:

$$\frac{u_{n-\tau_n}}{f(u_{n-\tau_n})} \le M+1,\ n \ge n_4$$

That is $u_{n-\tau_n} \le (M+1)f(u_{n-\tau_n}) + 1$, $n \ge n_4$ and so from (13) we get:

$$a_n\Delta u_n + u_{n-\tau_n} < 0,\ n \ge n_4 \quad (16)$$

In particular, from (16) for a subsequence (a_{n_k}) satisfying the condition (ii), we have:

$$u_{n_k+1} - u_{n_k} + u_{n_k-\tau_{n_k}} \le a_{n_k}(u_{n_k+1} - u_{n_k}) + u_{n_k-\tau_{n_k}} < 0$$

for k sufficiently large, which implies that $0 < u_{n_k+1} + (u_{n_k-\tau_{n_k}} - u_{n_k}) < 0$ for all large k.

This is a contradiction. The case that (Δu_n) is eventually positive can be treated in a similar fashion and so the proof of Theorem 3 is completed.

3. CONCLUSION

This study presents the design and implementation of the sufficient conditions for the oscillation of solutions the nonlinear difference equation of the form (01).

We believe that the present studies can be useful for the oscillation behavior of solutions of second order linear and nonlinear damped difference equations of the following forms:

$$\Delta(a_n\Delta u_n) + b_n\Delta u_n + c_n u_{n+1} = 0,\ \text{n=0,1,2,...}$$

$$\Delta(a_n\Delta u_n) + b_n\Delta u_n + c_n f(u_{n+1}) = 0$$

$$\Delta(a_n\varphi(u_n)\Delta u_n) + b_n\Delta u_n + c_n u_{n+1} = 0$$

$$\Delta(a_n\varphi(u_n)\Delta u_n) + b_n\Delta u_n + c_n f(u_{n+1}) = 0$$

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