COMPOSITE MODELS USED IN ACTUARIAL PRACTICE

Ştefănuţ CIOCHINĂ

Dunarea de Jos University, Galati, Romania (Stefanut.Ciochina@ugal.ro)

DOI: 10.19062/1842-9238.2019.17.2.8

Abstract: *In this paper, a summary of the composite models developed for use in actuarial practice is presented. We refer in extensive detail to the first composite model introduced in 2005 by Cooray and Ananda [3] which was then generalized by Scollnik [11] in 2007. The main features identified for these models were the density function, the cumulative distribution function, and the n-th order initial moment. We also look into some different variations of these composite models such as: Gamma - Pareto, Weibull - Pareto and Exponential - Pareto models.*

Keywords: *composite model, lognormal distribution, Pareto distribution, Gamma distribution, Weibull distribution, Exponential distribution, parameter estimation.*

1. INTRODUCTION

The modeling of claims data is a major challenge in the construction of applications in general insurance [2]. Insurance companies recorded in time losses that emerge from a combination of moderate and large claims [5]. The modeling of big losses is done in practice with the Pareto distribution. On the other hand, when losses consist of smaller values with high frequencies and larger losses with low frequencies [5], we use the lognormal distribution or Weibull distribution. However in [5], it is underlined that Pareto fits well the tail, but on the other hand, lognormal and Weibull distributions produce an overall good fit but fit badly the tail. Several works have introduced composite models for insurance loss data modeling [1, 6]. Cooray and Ananda [3] in 2005 were the first to open the way for research into composite models using a longnormal distribution to a certain threshold and then the Pareto distribution. Then in 2007 Scollnik [11] generalized the model proposed by Cooray and Ananda proposing two other composite models.

Insurance companies use data on the payment of positive claims. Their distribution often has a high upper tail [3]. Therefore, in the literature, an usual choise is the lognormal distribution or Pareto distribution to model such a data set (see FIG. 1.) [3]. In order to better capture the situations encountered in practice in one model, Cooray and Ananda introduces a composite model that uses lognormal density to a certain threshold and then Pareto density (FIG. 1.). Scollnik (2007) generalizes the proposed composite model [3] and introduces two new composite models (FIG. 2). All these models will be detailed in the next section.

The second section of this article is dedicated to the presentation of the two types of composite models, Cooray and Ananda [3] and Scollnik respectively [11]. For these models, the main features are mentioned: the density function, the cumulative distribution function and the *n-th* order initial moments. It is also presented the drawback of the model proposed by Cooray and Ananda, drawback emphasized by Scollnik in [11].

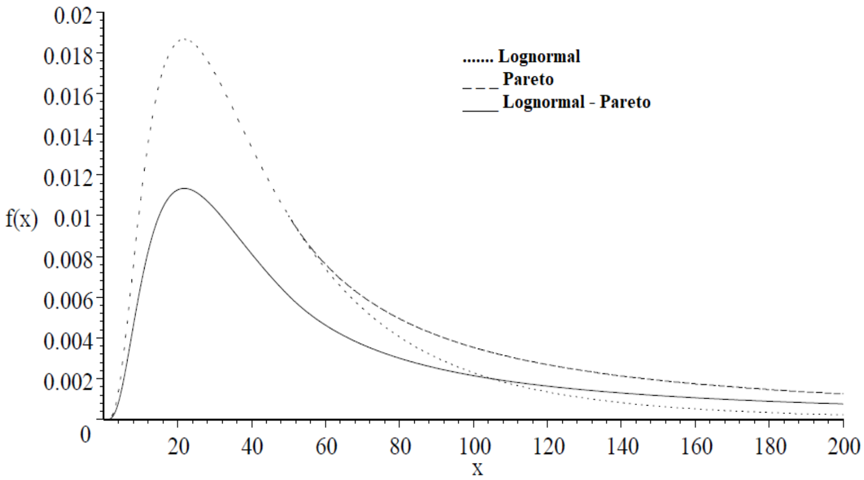

FIG 1. The composite lognormal-Pareto, Cooray and Ananda, model ($\theta = 50, \alpha = 0.5$) [3]

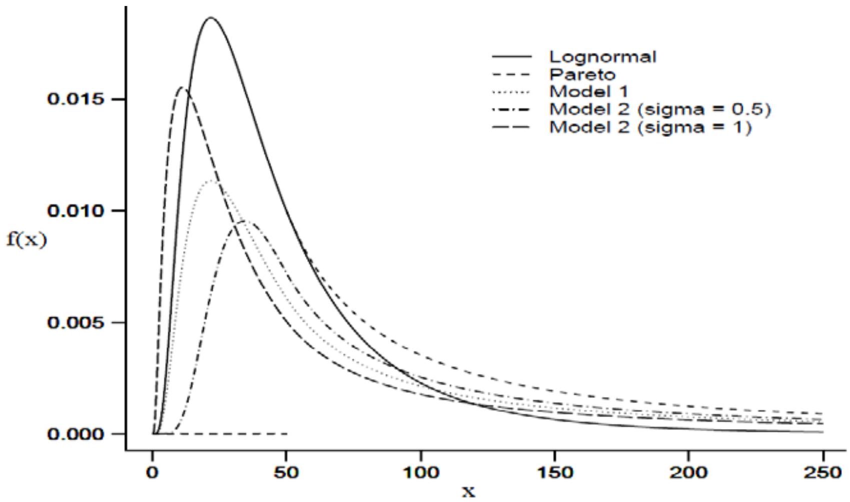

FIG 2. The composite lognormal-Pareto, Scollnik, model ($\theta = 50, \alpha = 0.5$) [11]

In the third section we present the main features of the following particular composite models: Gamma - Pareto, Weibull - Pareto, Exponential - Pareto models. Also, in section four we present a method for the parameter estimation.

2. FIRST COMPOSITE MODELS

2.1. Cooray and Ananda's model

The first lognormal-Pareto composite was developed by Cooray and Ananda (2005) [3] to model insurance payments. Cooray and Ananda construct the composite model considering a random variable X with probability density function:

$$f(x) = \begin{cases} cf_1(x), & 0 < x \leq \theta, \\ cf_2(x), & \theta < x < \infty, \end{cases} \tag{2.1}$$

where $f_1(x)$ and $f_2(x)$ are the lognormal and, respectively, Pareto probability density functions given in (2.2) and (2.3), and c is a normalizing constant.

$$f_1(x) = \frac{(2\pi)^{-1/2}}{x\sigma} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right), \quad x > 0 \tag{2.2}$$

$$f_2(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \qquad x > \theta \tag{2.3}$$

where $\theta, \mu, \sigma, \alpha$ are unknown parameters with the conditions: $\theta > 0, \sigma > 0, \alpha > 0$ and $\mu \in R$.

Imposing the conditions of continuity and differentiability at θ:

$$f(\theta - 0) = f(\theta + 0) \text{ and } f'(\theta - 0) = f'(\theta + 0), \tag{2.4}$$

they rewritten (2.1) in the form:

$$f(x) = \begin{cases} \frac{\alpha\theta^\alpha}{\left(1+\Phi(k)\right)x^{\alpha+1}} \exp\left\{-\frac{\alpha^2}{2k^2} ln^2\left(\frac{x}{\theta}\right)\right\}, & 0 < x \le \theta, \\ \frac{\alpha\theta^\alpha}{\left(1+\Phi(k)\right)x^{\alpha+1}}, & \theta < x < \infty, \end{cases} \tag{2.5}$$

where Φ (.) is the cumulative distribution function of the standard normal distribution. Also the value of the constant k is given by the positive solution of the equation $\exp(-k^2) = 2\pi k^2$. Thus the numerical value obtained for the constant k can be approximated by 0.372238898. Cooray and Ananda (2005) [3] also noticed that $\alpha\sigma = k$ and $c = 1/\left(1 + \Phi(k)\right)$. The conditions imposed in (4) ensure that they have a smooth probability density function. They also reduce the unknown parameters from four to two, $\theta > 0$ and $\alpha > 0$.

In [3] it is shown that the cumulative distribution function of the composite model is:

$$F(x) = \begin{cases} \frac{1}{\left(1+\Phi(k)\right)} \Phi\left((\alpha/k)\ln(x/\theta) + k\right), 0 < x \le \theta, \\ 1 - \frac{1}{\left(1+\Phi(k)\right)}(\theta/x)^\alpha, \qquad \theta < x < \infty, \end{cases} \tag{2.6}$$

For the composite lognormal-Pareto model (2.1), Cooray and Ananda show that the *n-th* moment is given by:

$$E(X^n) = \frac{\theta^n}{1+\Phi(k)}\left\{\Phi(k - kn/\alpha)\exp\left[\frac{1}{2}\left(\frac{k}{\alpha}\right)^2 (n^2 - 2\alpha n)\right] + \frac{\alpha}{\alpha - n}\right\}, n < \alpha \tag{2.7}$$

Cooray and Ananda [3] show that their proposed model can be applied by actuaries who encounter smaller data with higher frequencies as well as occasionally larger data with lower frequencies. Scollnik in [11] analyzes the model proposed by Corray and Ananda [3] and identifies a significant disadvantage.

Scollnik shows that the model proposed in [3] can be written:

$$f(x) = \begin{cases} \psi \dfrac{1}{\Phi(k)} f_1(x), & 0 < x \le \theta, \\ (1-\psi) f_2(x), & \theta < x < \infty, \end{cases} \tag{2.8}$$

where $\theta > 0, \alpha > 0$ and $f_1(x)$, $f_2(x)$ are the probability density functions given in (2.2) and (2.3). Also, $\Phi(k) = \Phi([\ln(\theta) - \mu]/\sigma)$. It results that:

$$\psi = \frac{\Phi(k)}{1+\Phi(k)} \approx 0.39215 \text{ and } 1 - \psi = \frac{1}{1+\Phi(k)} \approx 0.60785 \tag{2.9}$$

In [11] it is shown that the composite model proposed by Cooray and Ananda with fixed and a priori known mixing weights ψ and 1- ψ is restrictive. The theoretical model can be applied to any data set, but in order to obtain an optimal prediction, it is necessary to analyze in advance the set of data from practical activities.

2.2. Scollnik's models

2.2.1. The first composite Scollnik model

Scollnik [11] tried to eliminate the shortcomings of the Cooray and Ananda model by proposing a composite model like a longnormal truncated and Pareto. Thus, he rewroted the probability density function given in (2.1) in the form:

$$f(x) = \begin{cases} r f_1^*(x), & 0 < x \le \theta, \\ (1-r) f_2^*(x), & \theta < x < \infty, \end{cases} \tag{2.10}$$

where $0 \le r \le 1$, $f_1^*(x)$ and $f_2^*(x)$ represents the truncation of the density function $f_1(x)$ and $f_2(x)$, respectively. Also $f_1(x)$ and $f_2(x)$ are given by (2.2) and (2.3). So we get:

$$\begin{cases} f_1^*(x) = \dfrac{f_1(x)}{F_1(\theta)}, & 0 < x \le \theta, \\ f_2^*(x) = \dfrac{f_2(x)}{1 - F_2(x)}, & \theta < x < \infty, \end{cases} \tag{2.11}$$

A first observation regarding the composite model proposed by Scollnik is that the value of r is not constant like the value of c in the model proposed by Cooray and Ananda. Here the value of r belongs to the closed interval $[0,1]$ and is dependent on the particular values of θ, μ, σ and α.

In [17], using the continuity of function (2.10) at θ, is obtained:

$$f(\theta - 0) = f(\theta + 0) \Rightarrow r = \frac{f_2(\theta)F_1(\theta)}{f_2(\theta)F_1(\theta) + f_1(\theta)\big(1 - F_2(\theta)\big)}, \tag{2.12}$$

while from the condition of differentiability in θ is obtained:

$$f'(\theta - 0) = f'(\theta + 0) \Rightarrow r = \frac{f_2'(\theta)F_1(\theta)}{f_2'(\theta)F_1(\theta) + f_1'(\theta)\big(1 - F_2(\theta)\big)} \tag{2.13}$$

In [17] the expressions for the cumulative distribution function and for the *n–th* initial moment of the density function (2.10) are calculated. Thus is obtained:

$$F(x) = \begin{cases} r\dfrac{F_1(x)}{F_1(\theta)}, & 0 < x \le \theta, \\ r + (1-r)\dfrac{F_2(x) - F_2(\theta)}{1 - F_2(\theta)}, & \theta < x < \infty, \end{cases} \tag{2.14}$$

and

$$E_n(f) = rE_n(f_1^*) + (1-r)E_n(f_2^*). \tag{2.15}$$

In [13], the advantages of this type of model are presented compared to the non-truncated model.

2.2.2. The second Scollnik model

The second composite model proposed by Scollnik in [11] uses the truncated lognormal distribution for values less than the θ threshold value, and for values greater than the threshold value, uses the generalized Pareto distribution. Thus in [11] it uses the generalized version of Pareto distribution whose density function writes in the form:

$$f_2(x) = \frac{\alpha(\alpha\beta)^\alpha}{(\alpha\beta - \theta + x)^{\alpha+1}}, \qquad x > \theta, \tag{2.16}$$

where $\theta > 0, \alpha > 0$ and $\beta > 0$. If we denote $\gamma = \alpha\beta - \theta$ then the distribution function can be written:

$$f_2(x) = \frac{\alpha(\gamma + \theta)^\alpha}{(\gamma + x)^{\alpha+1}}, \qquad x > \theta, \tag{2.17}$$

where $\theta > 0, \alpha > 0$ and $\gamma > -\theta$. In conclusion, the new composite model has the density function given by the expression:

$$f(x) = \begin{cases} r\dfrac{f_1(x)}{F_1(x)}, & 0 < x < \theta, \\ (1-r)\dfrac{\alpha(\gamma+\theta)^\alpha}{(\gamma + x)^{\alpha+1}}, & \theta < x < \infty, \end{cases} \tag{2.18}$$

where $\theta > 0, \alpha > 0, \gamma > -\theta$ and $r \in [0,1]$. In [17], imposing the conditions of continuity and differentiability at the point θ the expressions for α and *r* were calculated. That's how they got:

$$\alpha + 1 = -\frac{(\gamma + \theta)f_1^{'}(\theta)}{f_1(\theta)}, \tag{2.19}$$

and

$$r = \frac{\alpha f_1^{'}(\theta)F_1(\theta)}{\alpha f_1^{'}(\theta)F_1(\theta) - (\alpha+1)f_1^2(\theta)}, \tag{2.20}$$

Also, in [17], the cumulative distribution function is calculated for this composite model:

$$F(x)=\begin{cases} r\dfrac{F_1(x)}{F_1(\theta)}, & 0<x\le\theta,\\ 1-(1-r)\left(\dfrac{\gamma+\theta}{\gamma+x}\right)^{\alpha}, & \theta<x<\infty,\end{cases} \tag{2.21}$$

and *n-th* order initial moment:

$$E_n(f)=rE_n(f_1^*)+(1-r)\alpha\sum_{k=0}^{n}\binom{n}{k}\frac{(-\gamma)^{n-k}(\gamma+\theta)^k}{\alpha-k},\qquad \alpha>n, \tag{2.22}$$

3. FURTHER COMPOSITE MODELS

3.1. Composite Gamma – Pareto models

3.1.1. The first composite Gamma - Pareto model

In [17], the composite model Gamma - Type II Pareto is developed following the construction steps shown in (2.18). For the development of the composite model, we refer to the form of the Gamma function $\Gamma(\nu)=\int_0^\infty x^{\nu-1}e^{-x}dx$ and to $\Gamma(\nu,t)=\int_0^t x^{\nu-1}e^{-x}dx,\ \nu,t>0$, the incomplete Gamma function.

Thus, in [17], the form of the density function for the composite model Gamma-Type II Pareto is given as:

$$f(x)=\begin{cases} r\dfrac{\beta^{\delta}}{\Gamma(\delta,\beta\theta)}x^{\delta-1}e^{-\beta x}, & 0<x<\theta,\\ (1-r)\dfrac{\alpha(\gamma+\theta)^{\alpha}}{(\gamma+x)^{\alpha+1}}, & \theta<x<\infty,\end{cases} \tag{3.1}$$

where $\beta,\delta,\alpha,\theta>0,\gamma>-\theta$ and $r\in[0,1]$. Imposing the conditions of continuity and differentiability at the point θ we obtain:

$$\alpha+1=\frac{(\gamma+\theta)(\beta\theta-\delta+1)}{\theta}, \tag{3.2}$$

and

$$r=\frac{\alpha(\beta\theta-\delta+1)\Gamma(\delta,\beta\theta)}{\alpha(\beta\theta-\delta+1)\Gamma(\delta,\beta\theta)+(\alpha+1)(\beta\theta)^{\delta}e^{-\beta\theta}}, \tag{3.3}$$

Using formula (2.21), in [17], is the form of the cumulative distribution function for this composite model:

$$F(x)=\begin{cases} r\dfrac{\Gamma(\delta,\beta x)}{\Gamma(\delta,\beta\theta)}, & 0<x\le\theta,\\ 1-(1-r)\left(\dfrac{\gamma+\theta}{\gamma+x}\right)^{\alpha}, & \theta<x<\infty,\end{cases} \tag{3.4}$$

and also using formula (2.22), the *n-th* order initial moment results as:

$$E_n(f) = r\frac{\Gamma(n+\delta,\beta\theta)}{\beta^n\Gamma(\delta,\beta\theta)} + (1-r)\alpha\sum_{k=0}^{n}\binom{n}{k}\frac{(-\gamma)^{n-k}(\gamma+\theta)^k}{\alpha-k},\ \alpha > n, \tag{3.5}$$

In [17], it is underlined that if $\delta = n$ is a positive integer, then the function value $\Gamma(n,.)$ can be written in recursive form:

$$\Gamma(n+1,x) = n\Gamma(n,x) - x^n e^{-x}, \qquad n \geq 1, x > 0, \tag{3.6}$$

with starting value $\Gamma(1,x) = 1 - e^{-x}, x > 0$.

3.1.2. The second composite Gamma – Pareto model

The composite Gamma – Pareto model, developed in [17], results from the composite Gamma – Type II Pareto model for $\gamma = 0$. From this we can obtain the form of the density function for the composite Gamma-Pareto model:

$$f(x) = \begin{cases} r\dfrac{\beta^\delta}{\Gamma(\delta,\beta\theta)}x^{\delta-1}e^{-\beta x}, & 0 < x < \theta, \\ (1-r)\dfrac{\alpha\theta^\alpha}{x^{\alpha+1}}, & \theta < x < \infty, \end{cases} \tag{3.7}$$

where $\alpha > 0, \beta > 0, \delta > 0, \theta > 0$ and $r \in [0,1]$. The parameters α and *r* were calculated by imposing the conditions of continuity and differentiability for density function (3.7):

$$\alpha = \beta\theta - \delta, \tag{3.8}$$

and

$$r = \frac{(\beta\theta-\delta)\Gamma(\delta,\beta\theta)}{(\beta\theta-\delta)\Gamma(\delta,\beta\theta) + (\beta\theta)^\delta e^{-\beta\theta}}, \tag{3.9}$$

For the model presented in this paragraph, in [17], it is observed that after applying the conditions (3.8) and (3.9) the number of parameters can be reduced from five to three.

If it is attempted to reduce the number of these parameters by using the second derivative, an impossible condition is reached, $\beta\theta^2 = 0$.

The cumulative distribution function, shown in [17], is given by:

$$F(x) = \begin{cases} r\dfrac{\Gamma(\delta,\beta x)}{\Gamma(\delta,\beta\theta)}, & x \leq \theta, \\ 1-(1-r)\left(\dfrac{\theta}{x}\right)^\alpha, & \theta < x < \infty, \end{cases} \tag{3.10}$$

and *n-th* order initial moment is:

$$E_n(f) = r\frac{\Gamma(n+\delta,\beta\theta)}{\beta^n\Gamma(\delta,\beta\theta)} + (1-r)\frac{\alpha\theta^n}{\alpha-n},\ \alpha > n, \tag{3.11}$$

3.2. Composite Weibull – Pareto models

3.2.1. The first composite Weibull – Pareto model

A first composite Weibull-Pareto model was developed by Ciumara (2006). This composite model is built based on the model presented by Cooray and Ananda in (2.1). In [10], a comparative study is made between the two composite distributions, longnormal - Pareto and Weibull - Pareto respectively. In comparison, the density functions, the cumulative distribution functions and the *n-th* order initial moment are discussed. Thus, in (1) they considered:

$$f_1(x) = \frac{\beta}{\gamma^\beta} x^{\beta-1} exp\left(-\left(\frac{x}{\gamma}\right)^\beta\right), x > 0, \gamma > 0, \beta > 1, \tag{3.12}$$

and

$$f_2(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, x > \theta, \theta > 0, \alpha > 0, \tag{3.13}$$

Thus, in [10], the density function for Weibull-Pareto composite distribution is obtained:

$$f(x) = \begin{cases} \dfrac{(t_0+1)^2}{(t_0+2)} \dfrac{\beta}{x} \left(\dfrac{x}{\theta}\right)^\beta exp\left(-(t_0+1)\left(\dfrac{x}{\theta}\right)^\beta\right), 0 < x \le \theta, \\ \dfrac{t_0(t_0+1)}{t_0+2} \dfrac{\beta}{x} \left(\dfrac{\theta}{x}\right)^{\beta t_0}, \qquad \theta < x < \infty, \end{cases} \tag{3.14}$$

The value of the constant t_0 was calculated by imposing the conditions of continuity and differentiability of the density function on $(0, \infty)$. This was approximated, in [10], by $t_0 \approx 0.34998$. Also, the number of parameters is reduced from four to two. The other two parameters are expressed using the relationships $\alpha = \beta t_0$ and $\gamma = \theta(t_0+1)^{\frac{1}{\beta}}$. Thus, the constant *c* of the density function definition is, in [10], $c = \frac{t_0+1}{t_0+2}$.

The cumulative distribution function for the Weibull – Pareto composite model is given in [10]:

$$F(x) = \begin{cases} \dfrac{t_0+1}{t_0+2}\left[1 - exp\left(-(t_0+1)\left(\dfrac{x}{\theta}\right)^\beta\right)\right], 0 < x \le \theta, \\ 1 - \dfrac{t_0+1}{t_0+2}\left(\dfrac{\theta}{x}\right)^{\beta t_0}, \qquad \theta < x < \infty, \end{cases} \tag{3.15}$$

The *n-th* order initial moments are given in [10]:

$$E_n(f) = \frac{t_0+1}{t_0+2}\theta^n \left[(t_0+1)^{-\frac{n}{\beta}} \Gamma\left(\frac{n}{\beta}+1, t_0+1\right) + \frac{\beta t_0}{\beta t_0 - n}\right] \tag{3.16}$$

for $n < \beta t_0$.

3.2.2. The second composite Weibull – Pareto model

The second composite Weibull-Pareto model is the one developed in [14] and [17]. It is a model built on the composite model described in (18). Thus, f_1 represents the Weibull density function, and f_2 represents the Type II Pareto density function.

Under these conditions, the Weibull - Type II Pareto composite model has the density function given by:

$$f(x)=\begin{cases} r\dfrac{1}{1-e^{-(\theta/\tau)^{\beta}}}\dfrac{\beta}{\tau^{\beta}}x^{\beta-1}e^{-(x/\tau)^{\beta}}, & 0<x<\theta, \\ (1-r)\dfrac{\alpha(\gamma+\theta)^{\alpha}}{(\gamma+x)^{\alpha+1}}, & \theta<x<\infty, \end{cases} \tag{3.17}$$

where the parameters $\beta,\tau,\alpha,\theta>0,\gamma>-\theta$ and $r\in[0,1]$. Imposing the conditions of continuity and differentiability on $(0,\infty)$, in [17], it is shown that:

$$\alpha+1=\frac{\gamma+\theta}{\theta}\left[\beta(\theta/\tau)^{\beta}-\beta+1\right], \tag{3.18}$$

and

$$r=\frac{\alpha\left[\beta(\theta/\tau)^{\beta}-\beta+1\right]\left[e^{(\theta/\tau)^{\beta}}-1\right]}{\alpha[\beta(\theta/\tau)^{\beta}-\beta+1]\left[e^{(\theta/\tau)^{\beta}}-1\right]+(\alpha+1)\beta(\theta/\tau)^{\beta}}, \tag{3.19}$$

Thus, the number of unknown parameters has been reduced from six to four. Also, in [17], it is shown that if it is still desired to reduce the number of unknown parameters the condition obtained is:

$$\begin{aligned} r\beta(\theta/\tau)^{\beta}\,&\frac{(\beta-1)(\beta-2)-3\beta(\beta-1)(\theta/\tau)^{\beta}+\beta^{2}(\theta/\tau)^{2\beta}}{\theta^{3}\left[e^{(\theta/\tau)^{\beta}}-1\right]}= \\ &=(1-r)\alpha(\alpha+1)(\alpha+2)\frac{1}{(\gamma+\theta)^{3}}, \end{aligned} \tag{3.20}$$

where $\gamma=\frac{\beta^{2}\theta^{\beta+1}}{(1-\beta)(\tau^{\beta}+\beta\theta^{\beta})}$.

The cumulative distribution function for the Weibull-Type II Pareto composite model in [17] is:

$$F(x)=\begin{cases} r\dfrac{1-e^{-(x/\tau)^{\beta}}}{1-e^{-(\theta/\tau)^{\beta}}}, & 0<x\le\theta, \\ 1-(1-r)\left(\dfrac{\gamma+\theta}{\gamma+x}\right)^{\alpha}, & \theta<x<\infty, \end{cases} \tag{3.21}$$

Here it was used that $F_1(x)=1-e^{-(x/\tau)^{\beta}}$.

For the composite Weibull - Type II Pareto the *n-th* order initial moment is given in [17]:

$$E_n(f) = r\frac{\tau^n}{1-e^{-(\theta/\tau)^\beta}}\Gamma\left(\frac{n}{\beta}+1,(\theta/\tau)^\beta\right) + (1-r)\alpha\sum_{k=0}^{n}\binom{n}{k}\frac{(-\gamma)^{n-k}(\gamma+\theta)^k}{\alpha-k}, \qquad \alpha > n, \tag{3.22}$$

3.2.3. The third composite Weibull – Pareto model

The third composite Weibull - Pareto model, developed in [17], represents a particular case of the composite Weibull - Type II Pareto model, for $\gamma = 0$. This gives the density function for the composite Weibull-Pareto model:

$$f(x) = \begin{cases} r\dfrac{1}{1-e^{-(\theta/\tau)^\beta}}\dfrac{\beta}{\tau^\beta}x^{\beta-1}e^{-(x/\tau)^\beta}, & 0 < x < \theta, \\ (1-r)\dfrac{\alpha\theta^\alpha}{x^{\alpha+1}}, & \theta < x < \infty, \end{cases} \tag{3.23}$$

where $\alpha, \beta, \tau, \theta > 0$ and $r \in [0,1]$. After applying the conditions of continuity and differentiability of the density function (3.23), it is obtained in [17]:

$$\alpha = \beta(\theta/\tau)^\beta - \beta + 1, \tag{3.24}$$

and

$$r = \frac{\alpha\left[e^{(\theta/\tau)^\beta} - 1\right]}{\alpha\left[e^{(\theta/\tau)^\beta} - 1\right] + \beta(\theta/\tau)^\beta}, \tag{3.25}$$

And this time, due to the application of conditions (3.26) and (3.27), the reduction of unknown parameters is from five to three. In [17], the observation is made that if it is attempted to reduce the number of unknown parameters to two it follows that $\beta^2\theta^{\beta+1} = 0$, which leads to $\beta = 0$, which is impossible, or $\theta = 0$ in which case the proposed model turns into the classic Pareto.

Also, in [17], we get the expressions for the cumulative distribution function:

$$F(x) = \begin{cases} r\dfrac{1-e^{-(x/\tau)^\beta}}{1-e^{-(\theta/\tau)^\beta}}, & 0 < x \le \theta, \\ 1-(1-r)\left(\dfrac{\theta}{x}\right)^\alpha, & \theta < x < \infty, \end{cases} \tag{3.26}$$

and *n-th* order initial moment for the composit Weibull – Pareto model:

$$E_n(f) = r\frac{\tau^n}{1-e^{-(\theta/\tau)^\beta}}\Gamma\left(\frac{n}{\beta}+1,(\theta/\tau)^\beta\right) + (1-r)\frac{\alpha\theta^n}{\alpha-n}, \ \alpha > n, \tag{3.27}$$

3.3. Composite Exponential – Pareto models

3.3.1. The first composite Exponential – Pareto model

The composite exponential - Pareto model was developed in [16] according to the model described in [3]. Thus, in the model constructed in (2.1), f_1 is considered to be the exponential density and f_2 the Pareto density. In conclusion, in [16], it is considered:

$$f_1(x) = \lambda e^{-\lambda x}, x > 0, \tag{3.28}$$

$$f_2(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, x > \theta, \tag{3.29}$$

where $\lambda > 0, \alpha > 0, \theta > 0$ are unknown parameters.

Following the application of the constraints of continuity and differentiability of the density function it is obtained in [16]:

$$\begin{cases} \lambda e^{-\lambda\theta} = \dfrac{\alpha}{\theta}, \\ \lambda^2 e^{-\lambda\theta} = \dfrac{\alpha(\alpha+1)}{\theta^2}, \end{cases} \tag{3.30}$$

The authors get:

$$\begin{cases} \lambda\theta = 1.35 \\ \alpha = 0.35 \end{cases} \tag{3.31}$$

After the restriction system was resolved, it was possible to reduce the unknown parameters from three to one. Also, imposing the condition $\int_0^\infty f(x)dx = 1$ the normalization constant is obtained:

$$c = \frac{1}{2 - e^{-\lambda\theta}} = 0.574, \tag{3.32}$$

The density function for the composite Exponential - Pareto model can be written as:

$$f(x) = \begin{cases} \dfrac{0.775}{\theta} e^{-\frac{1.35x}{\theta}}, & 0 < x \le \theta, \\ 0.2 \dfrac{\theta^{0.35}}{x^{1.35}}, & \theta < x < \infty, \end{cases} \tag{3.33}$$

And the cumulative distribution function, in [16], is given as:

$$F(x) = \begin{cases} 0.574\left(1 - e^{-\frac{1.35x}{\theta}}\right), 0 < x \le \theta, \\ 1 - 0.574\left(\dfrac{\theta}{x}\right)^{0.35}, \theta < x < \infty, \end{cases} \tag{3.34}$$

3.3.2. The second composite Exponential – Pareto model

The second composite Exponential – Pareto model, developed in [15], is built on the model (2.18). A generalized Pareto distribution is used, in [15], above the threshold θ. Thus, the second Exponential – Pareto composite model has the density function:

$$f(x) = \begin{cases} r\dfrac{\lambda e^{-\lambda x}}{1 - e^{-\lambda\theta}}, & 0 < x \le \theta, \\ (1-r)\dfrac{\alpha(\gamma+\theta)^\alpha}{(\gamma+x)^{\alpha+1}}, \theta < x < \infty, \end{cases} \tag{3.35}$$

where the parameters $\lambda, \alpha, \theta > 0, \gamma > -\theta$ and $r \in [0,1]$. In [15], applying the conditions of continuity and differentiability on $(0, \infty)$, the following conditions are obtained:

$$\alpha + 1 = \lambda(\gamma + \theta), \tag{3.36}$$

and

$$r = \frac{\alpha\left(1 - e^{-\lambda\theta}\right)}{\alpha + e^{-\lambda\theta}}, \tag{3.37}$$

It is noted that following the application of the two conditions, continuity and differentiability, the number of unknown parameters decreased from five to three. In [15] trying to reduce the number of unknown parameters using a second derivative requirement yields:

$$\frac{r}{1 - e^{-\lambda\theta}} \lambda^3 e^{-\lambda\theta} = (1 - r)\alpha(\alpha + 1)(\alpha + 2)\frac{1}{(\gamma + \theta)^3}, \tag{3.38}$$

and using (3.36) and (3.37) the authors conclude $\alpha + 1 = \alpha + 2 \Leftrightarrow 0 = 1$, which is impossible.

The cumulative distribution function for the composite Exponential - Type II Pareto model is:

$$F(x) = \begin{cases} r\dfrac{1 - e^{-\lambda x}}{1 - e^{-\lambda x}}, 0 < x \leq \theta, \\ 1 - (1 - r)\left(\dfrac{\gamma + \theta}{\gamma + x}\right)^{\alpha}, \theta < x < \infty, \end{cases} \tag{3.39}$$

and the *n-th* order moment of the composite Exponential – Type II Pareto is given by, in [15]:

$$E_n(f) = r\frac{\Gamma(n + 1, \lambda\theta)}{\lambda^n(1 - e^{-\lambda\theta})} + (1 - r)\alpha \sum_{k=0}^{n} \binom{n}{k} \frac{(-\gamma)^{n-k}(\gamma + \theta)^k}{\alpha - k}, \alpha > n, \tag{3.40}$$

where Γ is the incomplete gamma function.

3.3.3. The third composite Exponential – Pareto model

The third Exponential - Pareto composite model, developed in [15], is designed according to the model (2.18). It's a composite Exponential-Pareto model define as a truncated Exponential and Pareto mixture with threshold value θ [15]. In other words, this model is a particular case obtained by taking, $\gamma = 0$, of the composite model presented in the previous section.

The third composite Exponential – Pareto model has the density function:

$$f(x)=\begin{cases} r\dfrac{\lambda e^{-\lambda x}}{1-e^{-\lambda\theta}}, & 0<x\le\theta,\\ (1-r)\dfrac{\alpha\theta^{\alpha}}{x^{\alpha+1}}, & \theta<x<\infty, \end{cases} \tag{3.41}$$

where $\lambda>0,\alpha>0,\theta>0$ and $r\in[0,1]$. Also, by imposing the conditions of continuity and differentiability on $(0,\infty)$ to the density function (3.41), one obtains as in [15]:

$$\alpha+1=\lambda\theta, \tag{3.42}$$

and

$$r=\frac{\alpha\left(1-e^{-\lambda\theta}\right)}{\alpha+e^{-\lambda\theta}}=\frac{\alpha\left(1-e^{-(\alpha+1)}\right)}{\alpha+e^{-(\alpha+1)}}, \tag{3.43}$$

In [15], it is shown that because of conditions (3.42) and (3.43) the number of unknown parameters is reduced from four to two. If it is still desired to reduce the number of unknown parameters one can use the second order derivative. This leads to:

$$\frac{r}{1-e^{-\lambda\theta}}\lambda^{3}e^{-\lambda\theta}=(1-r)\alpha(\alpha+1)(\alpha+2)\frac{1}{\theta^{3}} \tag{3.44}$$

But if in relation (3.44) we use relations (3.42) and (3.43) we are led to $\alpha+1=\alpha+2\Leftrightarrow 0=1$, which is impossible.

The cumulative distribution function for the third composite Exponential - Pareto is given in [15]:

$$F(x)=\begin{cases} r\dfrac{1-e^{-\lambda x}}{1-e^{-\lambda x}}, & 0<x\le\theta,\\ 1-(1-r)\left(\dfrac{\theta}{x}\right)^{\alpha}, & \theta<x<\infty, \end{cases} \tag{3.45}$$

And the initial *n-th* order moments are:

$$E_n(f)=r\frac{\Gamma(n+1,\lambda\theta)}{\lambda^{n}(1-e^{-\lambda\theta})}+(1-r)\frac{\alpha\theta^{n}}{\alpha-n},\alpha>n, \tag{3.46}$$

where Γ is the incomplete gamma function.

4. PARAMETER ESTIMATION

An important aspect is the necessity of estimating the unknown parameter θ. Many studies [7,8] use on the maximum likelihood method as the method of estimating parameter. Thus, consider the case of the composite model proposed by Scollnik (2007) whose density function is given by (2.18), with the real parameters $\delta_1,\delta_2,\delta_3,\ldots,\delta_s,\theta$, with $s\in N$ and $x_1\le x_2\le x_3\le\cdots\le x_n$ an ordered sample of data from the composite (2.18). In [16], it is specified that if the likelihood function is to be evaluated, it is necessary to know where the unknown parameter θ is placed in relation to that sample.

Assuming that the unknown parameter θ is placed $x_m \leq \theta \leq x_{m+1}$, then the likelihood function is given in [17]:

$$L(x_1, \dots, x_n, \delta_1, \dots \delta_n, \theta) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{m} r f_1^*(x_i) \prod_{i=m+1}^{n} (1-r) f_2^*(x_j)$$
$$= r^m (1-r)^{n-m} \prod_{i=1}^{m} f_1^*(x_i) \prod_{i=m+1}^{n} f_2^*(x_j). \quad (4.1)$$

Since the proposed method depends on m, which is not known exactly, in [17] the following algorithm is proposed:

Step 1. For each $m = 1,2,3, \dots, n-1$, evaluate $\widehat{\delta_1}, \dots, \widehat{\delta_s}, \hat{\theta}$ as solutions of the system:

$$\begin{cases} \frac{\partial \ln L}{\partial \delta_i} = 0, \quad i = 1,2,3, \dots, s, \\ \frac{\partial \ln L}{\partial \theta} = 0 \end{cases} \quad (4.2)$$

If $\hat{\theta}$ is located between $x_m \leq \hat{\theta} \leq x_{m+1}$ then $\hat{\theta}$ is the maximul likelihood estimator. If not, go to step 2

Step 2. If the system (4.2) has no solution then we are in one of two situations $m = n$ or $m = 0$. In this case, in [17] it is recommended to use one of the functions f_1 or f_2 for the likelihood function.

Unknown parameter estimation using this method can be implemented on a computing platform as shown in [12]. This may lead to reduced work-time in terms of system solving (4.1).

CONCLUSIONS

In this paper we presented a summary of the characteristics of the main composite models used in the processing of statistical data in actuarial. We begin the work with the basic composite models, Cooray and Ananda (2005) and Scollnik (2007), and then introduce the particular Gamma-Pareto, Weibull-Pareto and Exponential-Pareto models. Depending on the distribution of the data to be processed, one or other of the models presented may be applied.

There are other composite models studied in the literature like, e.g.: composite truncation models [13], composite lognormal – Burr [1], composite Stoppa models [2], inverse Weibull composite models [4], composite lognormal – Pareto model with random threshold [9].

REFERENCES

[1] S.A. Abu Bakar, N.A. Hamzah, M. Maghsoudi and S. Nadarajah, *Modeling loss data using composite models.* Insurance: Mathematics and Economics 61, 2015, 146 – 154;

[2] E. Calderin-Ojeda and C.F. Kwok, *Modeling claims data with composite Stoppa models*, Scand. Actuar. J. 9, 2016, 817 – 836;

[3] K. Cooray and M.A. Ananda, *Modeling actuarial data with a composite Lognormal-Pareto model.* Scand. Actuar. J. 5, 2005, 321–334;

[4] K. Cooray, S. Gunasekera and M.A. Ananda, *Weibull and inverse Weibull composite distribution for modeling reliability data*, Model Assisted Statistics and Applications 5, 2010, 109–115;

[5] Y. Dominicy and C. Sinner, *Distributions and composite models for size-type data*, 2017, http://dx.doi.org/10.5772/66443;

[6] S. A. Klugman, H. H. Panjer and G. E. Willmot, *Loss Models From Data to Decisions.* New York: John Wiley & Sons, Inc., 1998;

[7] S. Nadarajah and S.A.A. Bakar, *New composite models for the Danish fire insurance data.* Scand. Actuar. J. 2012, 1–8;

[8] S. Nadarajah and S.A.A. Bakar, *CompLognormal: An R Package for Composite Lognormal Distributions.* The R Journal Vol. 5/2, 2013, 97 – 103;

[9] M. Pigeon and M. Denuit, *Composite Lognormal – Pareto model with random threshold,* Scand. Actuar. J. 3, 2011, 177–192;

[10] V. Preda and R. Ciumara, *On composite models: Weibull-Pareto and Lognormal-Pareto. A comparative study*. Rom. J. Econ. Forecast. 3, 2006, 32–46;

[11] D.P.M. Scollnik, *On composite Lognormal-Pareto models*. Scand. Actuar. J. 1, 2007, 20–33;

[12] D.P.M. Scollnik and C. Sun, *Modeling with Weibull – Pareto models.* North American Actuarial J. 16, 2, 2012, 260 – 272;

[13] S. Teodorescu, *On the truncated composite Lognormal – Pareto model.* Math Reports 12 (62), 1, 2010, 71 – 84;

[14] S. Teodorescu and E. Panaitescu, *On the truncated composite Weibull – Pareto model.* Math Reports 11 (61), 3, 2009, 259 – 273;

[15] S. Teodorescu and R. Vernic, *Some composite Exponential-Pareto models for actuarial prediction*. Rom. J. Econ. Forecast. 12, 2009, 82–100;

[16] S. Teodorescu and R. Vernic, *A composite Exponential – Pareto distribution*. An. St. Univ. “Ovidius” Constanta Ser. Mat. 14, 2006, 1, 99–108;

[17] S. Teodorescu and R. Vernic, *On composite Pareto model.* Math Reports 15 (65), 1, 2013, 11 – 29.