2. ADOMIAN DECOMPOSITION METHOD

In this section we give some brief and basic information about Adomian decomposition method. For much more information, we refer to [5,13,14].

Consider the differential equation

\[ Ly + Ry + Ny = g(t) \]  (1)

where \( L \) is the highest-order derivative which is assumed to be invertible, \( R \) is a linear differential operator of less order than \( L \), \( N \) is the nonlinear operator and \( g \) is the source term. If we apply the operator \( L^{-1} \) which is the inverse of the \( L \) to the equation (1), we get

\[ L^{-1}(Ly) = y = L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny). \]  (2)

Let us suppose the solution of the Eq.(1):

\[ y(t) = \sum_{n=0}^{\infty} y_n(t). \]  (3)

Besides that the nonlinear terms is obtained by

\[ Ny = \sum_{n=0}^{\infty} A_n \]  (4)

where \( A_n \) are Adomian polynomials which can be calculated from:
where $a_n$ are coefficients and $T_n$ are Chebyshev polynomials [16,17].

In fact, there are many orthogonal polynomials as Laguerre, Legendre etc. that we can use instead of Taylor polynomials. But, Tien [18] and Mahmoudi [19] showed that these modifications are not good enough as much as Chebyshev polynomials.

4. NUMERICAL EXAMPLES

In this section, we solve two Riccati equations to illustrate the phenomena. These problems are brand new and cannot be found in the literature.

**Example 1**

Consider the Riccati differential equation

$$y' - \frac{1}{2}y + y^2 = e^t, \quad y(0) = 1$$

which have the exact solution $y = e^{t^2}$.

**Solution:**

We proceed according to section 2. We have

$$L = \frac{d}{dt}R(y) = \frac{-y}{2}, \quad Ny = F(y) = y^2$$

and $g(t) = e^t$.

Constructing Adomian polynomials according to (5), we obtain:

$$A_0 = F(y_0) = y_0^2$$
$$A_1 = y_1 F'(y_0) = 2y_0y_1$$
$$A_2 = y_2 F'(y_0) + \frac{1}{2!}y_1^2 F''(y_0) = 2y_0y_1 + y_1^2$$

Writing the source function in Taylor series form for only 4 terms:

$$g(t) = e^t \approx 1 + t + \frac{t^2}{2} + \frac{t^3}{6}$$

Then we form the recursive relation as in (7):
\[ y_0 = f = y(0) + L^{-1}(1 + t + \frac{t^2}{2} + \frac{t^3}{6}) \]
\[ y_1 = -L^{-1}R(y_0) - L^{-1}(A_0) \quad (15) \]
yields
\[ y_{k+1} = -L^{-1}\left(R\left(y_k\right)\right) - L^{-1}(A_k), \quad k \geq 0 \]

\[ y_0 = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} \]
\[ y_1 = -\frac{t}{2} - \frac{3t^2}{4} - \frac{7t^3}{12} - \frac{5t^4}{16} - \frac{7t^5}{60} + \ldots \quad (16) \]
\[ y_2 = -\frac{t^2}{8} + \frac{17t^3}{24} + \frac{23t^4}{32} + \ldots \]
\[ \vdots \]

After doing much more calculations, we get the solution as:

\[ y_T(t) = \sum_{n=0}^{m} y_n = y_0 + y_1 + y_2 + \cdots y_m \quad (17) \]

where \( y_T(t) \) denotes the approximate solution computed by using Taylor series expansions. For \( m = 4 \) we obtain

\[ y_T(t) = 1 + 0.5t + 0.125t^2 + 0.020833t^3 + 0.002604t^4 + 0.000520t^5 \quad (18) \]

Now, we make the same calculations by using Chebyshev expansion which can be calculated as in [17,18]:

\[ g(t) = e^t \approx \frac{382}{384} + \frac{383}{384} t + \frac{208}{384} t^2 + \frac{68}{384} t^3 + \ldots \quad (19) \]

Again, we have now the recursive relation:

\[ y_0 = 1 + 0.9947916t + 0.9973958r^2 + 0.53125t^3 + \cdots \]
\[ y_1 = -0.5t - 0.746094t^2 - 1.32857r^3 - 1.19272t^4 + \cdots \quad (20) \]
\[ y_2 = -0.125t^2 - 0.207682t^3 - 0.352595r^4 + \cdots \]
\[ \vdots \]

By proceeding for \( m = 4 \) we get

\[ y_C(t) = 1 + 0.49947916t + 0.126301r^2 + 0.021991t^3 + 0.002697t^4 + 0.000601t^5 \quad (21) \]

For larger \( m \) more accurate results we get. Figure 1 and Figure 2 displays the errors for only \( m = 4 \). Table 1 shows the comparisons of these errors for \( m = 8 \).

**Example 2** Consider the Riccati differential equation

\[ y' - ty + y^2 = e^t, \quad y(0) = 1 \quad (22) \]

which have the exact solution \( y = e^t \).
Figure 3 and Figure 4 displays the absolute errors for only $m = 4$. We can see the difference even for small $m$. Table 2 shows the comparisons of these errors for $m = 8$.

**CONCLUSIONS**

In this study, we show that using Chebyshev polynomials is a good idea to improve the effectiveness of the Adomian decomposition method. We use Chebyshev expansions of the source term to obtain more accurate results. Figures enable us to see that the difference between the using both two methods by graphically. Tables are also given to show the variation of the absolute errors for larger approximation, namely for larger $m$. Maple 18 is used for calculations and sketching graphs.

### Table 1: Absolute errors for $m = 8$ for Example 1.

| $t$  | $y(t)$   | $|y - y_C|$          | $|y - y_T|$          |
|------|----------|---------------------|---------------------|
| 0.2  | 1.1051709| 8.054395E-10        | 6.762634E-8         |
| 0.4  | 1.2214028| 5.043533E-8         | 5.595943E-6         |
| 0.6  | 1.3498588| 3.766847E-7         | 1.939455E-5         |
| 0.8  | 1.4918247| 1.504646E-6         | 4.543433E-4         |

**Solution:**

We here have

$$L = \frac{d}{dt}, R(y) = \frac{-t}{2}, Ny = F(y) = y^2 \text{ and } g(t) = e^{c^t}. \quad (23)$$

We can easily compute the Taylor and Chebyshev series expansions of the source terms we need:

$$g_T(t) = e^{c^t} \approx 1 + t^2 + 0.5t^4 + 0.16666t^6 + 0.04166t^8 + \cdots \quad (24)$$

and

$$g_C(t) = e^{c^t} \approx 0.99997 + 1.0001t^2 + 0.49912t^4 + 0.17036t^6 + 0.034853t^8 + \cdots \quad (25)$$

Following the same procedure as in the previous example 1, we get the approximate solutions:

$$y_T(t) = 1 + 0.5t^2 + 0.125t^4 + 0.020833t^6 + 0.0026041t^8 + 0.0002604t^{10} + \cdots \quad (26)$$

and

$$y_C(t) = 1 + 0.49999998t^2 + 0.12500037t^4 + 0.02083112t^6 + 0.002610400t^8 + 0.0002514712t^{10} + \cdots$$
Table 2: Absolute errors for $m = 8$ for Example 2.

| $t$   | $y(t)$        | $|y - y_c|$        | $|y - y_T|$        |
|-------|---------------|-------------------|-------------------|
| 0.2   | 1.0202013     | 3.053732E-12      | 6.039393E-11      |
| 0.4   | 1.0832871     | 5.958308E-10      | 1.003932E-10      |
| 0.6   | 1.1972174     | 1.958503E-9       | 4.900392E-7       |
| 0.8   | 1.3771278     | 2.408753E-6       | 7.540059E-5       |

Figure 4: The errors $|y - y_c|$ for Example 2.

Modified Adomian Decomposition Method for Solving Riccati Differential Equations