FIXED POINT THEOREM FOR φ_M-GERAGHTY CONTRACTIONS

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Abstract: *In this paper we extend the result of Geraghty([4],[5]) about φ- contractions, starting from the papers of O.Popescu([7]) respectively A.Fulga, A. Proca ([2]). We introduce a new notation and we establish a fixed point theorem for such mapping in a complete metric space.*
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1. INTRODUCTION

Because of its importance in mathematics and specially in fixed point theory, a lot of authors ([6],[7],[8],[9]) gave generalizations of Banach contraction principle [1]. One of the most well-known generalizations is given by Geraghty[5].

In this paper, starting from [7] and [2], we introduce the notion of φ_M-Geraghty contraction and prove a fixed point theorem for φ_M-contractions, which generalized theorem (1.1).

Theoreme (1.1) Let (X,d) be a complete metric space and T:X→X be an operator. If T satisfies the following inequality:

$$d(Tx,Ty) \leq \varphi\big(d(x,y)\big) \cdot d(x,y), \forall x,y \in X, \tag{1.1}$$

where $\varphi: [0,\infty) \rightarrow [0,1)$ is a function which satisfies the condition

$$\lim_{n\rightarrow\infty} \varphi(t_n) = 1 \Rightarrow \lim_{n\rightarrow\infty} t_n = 0 \tag{1.2}$$

then T has a unique fixed point.

2. MAIN RESULTS

Definition (2.1) Let (X,d) be a metric space. A mapping T:X→X is said to be a φ_E-Geraghty contraction on (X,d) if there exists $\varphi \in \emptyset$ such that

$$d(Tx,Ty) \leq \varphi\big(E(x,y)\big)E(x,y), \forall x,y \in X,$$

where

$$E(x,y) = d(x,y) + |d(x,Tx) - d(y,Ty|$$

and $\emptyset$ denote the class of functions $\varphi: [0,\infty) \to [0,1)$ which satisfies the condition:

$$\lim_{n\to\infty} \varphi(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0 \qquad (2.1)$$

Theorem (2.1) Let (X,d) be a complete metric space and T:X→X be an φ_E-Geraghty contraction. Then T has a unique fixed point $x^* \in X$ and for all $x_0 \in X$ the sequence $\{T^n x_0\}$ is convergent to x^*.

Definition (2.2) Let (X,d) be a metric space. A mapping T:X→X is said to be a φ_M −Geraghty contraction on (X,d) if there exists $\varphi \in \emptyset$ such that

$$d(Tx,Ty) \le \varphi\big(M(x,y)\big)M(x,y) \qquad (2.2)$$

where

$$M(x,y) = \max\{d(x,y) + |d(x,Tx) - d(y,Ty)|; d(x,Tx) + |d(x,y) - d(y,Ty)|; \qquad (2.3)$$

$$d(y,Ty) + |d(x,y) - d(x,Tx)|; \frac{d(x,Ty) + d(y,Tx) + |d(x,Tx) - d(y,Ty)|}{2}\}$$

and $\emptyset$ denote the class of functions $\varphi: [0,\infty) \to [0,1)$which satisfies the condition: $\lim_{n\to\infty} \varphi(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0$.

Theorem(2.1) Let (X,d) be a complete metric space and T:X→X be φ_M-Geraghty contraction. Then T has a unique fixed point $x^* \in X$ and for all $x_0 \in X$ the sequence $\{T^n x_0\}$ is convergent to x^*.

Demonstration:

Let $x_0 \in X$, arbitrary, fixed, with $x_{n+1} = Tx_n = T^n x_0$, then x_0 is fixed point for T. We can suppose that $x_n \neq x_{n+1}$, for all natural *n* so it results $d(x_n, x_{n+1}) > 0, \forall n \in N$.

If we denote $d(x_n, x_{n+1}) = d_n$ and put x=x_n and y=x_{n+1} in (2.2)

$d(Tx,Ty) \le \varphi\big(M(x,y)\big) \cdot M(x,y),$ where φ:[0,∞) → [0,1) and

$[\varphi(t_n) \to 1] \Rightarrow t_n \to 0,$ we obtain

$$d(Tx_n,Tx_{n+1}) \le \varphi\big(M(x_n,x_{n+1})\big) \cdot M(x_n,x_{n+1}).$$

$$M(x_n,x_{n+1}) = \max\{d_n + |d_n - d_{n+1}|; d_n + |d_n - d_{n+1}|,$$

$$d_{n+1} + |d_n - d_n|, \frac{d(x_n,x_{n+2}) + d(x_{n+1},x_{n+1}) + |d_n - d_{n+1})|}{2}\Bigg\}$$

If $d_{n+1} > d_n \Rightarrow$

$$M(x_n,x_{n+1}) = max\left\{d_{n+1}, \frac{d(x_n,x_{n+2}) + d_{n+1} - d_n}{2}\right\}$$

But

$$\frac{d(x_n,x_{n+2})+d_{n+1}-d_n}{2} \le \frac{d_n+d_{n+1}+d_{n+1}-d_n}{2} = d_{n+1}$$

So we have $M(x_n,x_{n+1}) = d_{n+1}$.

From the assumption on the theorem we get

$d(x_{n+1},x_{n+2}) \le \varphi(d_{n+1}) \cdot d_{n+1}$, and we obtain
$d_{n+1} \le \varphi(d_{n+1}) < d_{n+1}$,

which is a contradiction.

If $d_n > d_{n+1}$, we have

$$M(x_n,x_{n+1}) = max\left\{2d_n - d_{n+1}, d_{n+1}, \frac{d(x_n,x_{n+2})+d_n-d_{n+1}}{2}\right\}$$

But $2d_n - d_{n+1} > d_{n+1}$, and

$$\frac{d(x_n,x_{n+2})+d_n-d_{n+1}}{2} \le \frac{d_n+d_{n+1}+d_n-d_{n+1}}{2} = d_n$$

$$2d_n - d_{n+1} = d_n + (d_n - d_{n+1}) > d_n.$$

So $M(x_n,x_{n+1}) = 2d_n - d_{n+1}$

From the assumption of the theorem, we get

$$d(x_{n+1},x_{n+2}) \le \varphi(2d_n - d_{n+1}) \cdot (2d_n - d_{n+1})$$

$$d_{n+1} \le \varphi(2d_n - d_{n+1})(2d_n - d_{n+1}) \tag{2.4}$$

$$d_{n+1} \le 2d_n - d_{n+1}$$

Therefore $d_n \ge d_{n+1}, \forall n \in \mathbb{N}$.

Let $d = \lim_{n\to\infty} d_n$ and we suppose that $d > 0$. Taking the limit as $n \to \infty$ in (2.4) we get

$$d \le \lim_{n\to\infty}[\varphi(2d_n - d_{n+1}) \cdot (2d_n - d_{n+1})] \le \lim_{n\to\infty}(2d_n - d_{n+1})$$

$$d \le \lim_{n\to\infty} \varphi(2d_n - d_{n+1}) \cdot d \le d$$

$$\Rightarrow \lim_{n\to\infty} \varphi(2d_n - d_{n+1}) = 1 \qquad \Rightarrow \lim_{n\to\infty}(2d_n - d_{n+1}) = 0 \quad \Rightarrow d = 0.$$

We prove now, that $\{x_n\}$ is a Cauchy sequence. Suppose, on the contrary, that there exists $\varepsilon > 0$ and $\{n(k)\},\{m(k)\} \subset \mathbb{N}$, $n(k) > m(k) > k$, such that

$$d\left(x_{n(k)}, x_{m(k)}\right) \geq \varepsilon, d\left(x_{n(k)-1}, x_{m(k)}\right) < \varepsilon, (\forall)k \in \mathbb{N}. \tag{2.5}$$

Using the triangle inequality and (2.5), we get:

$$\varepsilon \leq d\left(x_{n(k)}, x_{m(k)}\right) \leq d\left(x_{n(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{m(k)}\right). \tag{2.6}$$

Taking the limit as $k \to \infty$ in (2.6) and using (2.5) we obtain

$$\lim_{n\to\infty} d\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon \tag{2.7}$$

Also

$$\begin{aligned} &\left|d\left(x_{n(k)+1}, x_{m(k)+1}\right) - d\left(x_{n(k)}, x_{m(k)}\right)\right| \leq \\ &\leq d\left(x_{n(k)}, x_{n(k)+1}\right) + d\left(x_{m(k)}, x_{m(k)+1}\right), \\ &and \lim_{n\to\infty} d\left(x_{n(k)+1}, x_{m(k)+1}\right) = \varepsilon \end{aligned} \tag{2.8}$$

Putting $x = x_{n(k)-1}, y = x_{m(k)-1}$ in relation (2.2) we deduce

$$\varepsilon \leq d\left(x_{n(k)}, x_{m(k)}\right) \leq$$

$$\leq \varphi\left(M(x_{n(k)-1}, x_{m(k)-1})\right) \cdot M(x_{n(k)-1}, x_{m(k)-1}), (\forall)k \in \mathbb{N}. \tag{2.9}$$

where

$$\begin{aligned} &M(x_{n(k)-1}, x_{m(k)-1}) = \\ &\max\{d\left(x_{n(k)-1}, x_{m(k)-1}\right) + \left|d\left(x_{n(k)-1}, x_{n(k)}\right) - d(x_{m(k)-1}, x_{m(k)}\right|, \\ &d\left(x_{n(k)-1}, x_{n(k)}\right) + \left|d(x_{n(k)-1}, x_{m(k)-1}) - d(x_{m(k)-1}, x_{m(k)})\right|, \\ &\qquad d\left(x_{m(k)-1}, x_{n(k)}\right) + \left|d(x_{n(k)-1}, x_{m(k)-1}) - d(x_{n(k)-1}, x_{n(k)})\right|, \\ &\tfrac{1}{2}\left[d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)}) + \left|d(x_{n(k)-1}, x_{n(k)}) - d(x_{m(k)-1}, x_{m(k)})\right|\right]\} \end{aligned}$$

We observe that

$$\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon > 0 \tag{2.10}$$

Taking the limit as $k \to \infty$ in (2.9) and using (2.10) we obtain

$$\varepsilon \leq \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) \leq$$

$$\leq \lim_{k\to\infty} \left[\varphi\left(M(x_{n(k)-1}, x_{m(k)-1})\right) \cdot M(x_{n(k)-1}, x_{m(k)-1})\right] < \varepsilon,$$

and

$$\lim_{k\to\infty} \left(M(x_{n(k)-1}, x_{m(k)-1})\right) \cdot \varepsilon = \varepsilon,$$

so

$$\lim_{k\to\infty} \varphi\left(M(x_{n(k)-1}, x_{m(k)-1})\right) = 1 \Rightarrow$$

$$\Rightarrow \lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = 0,$$

which is a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d),$\{x_n\}$ is convergent to

$$x^* \in X \ and \ \lim_{k\to\infty} d(x_n, x^*) = 0 \qquad (2.11)$$

Finally, we will show that $x^* = Tx^*$. We put $x = x_n$ and $y = x^*$ in (2.2):

$$d(Tx_n, Tx^*) \leq \varphi\left(M(x_n, x^*)\right) \cdot M(x_n, x^*)$$

$$d(x_{n+1}, Tx^*) \leq \varphi\left(M^*(x_n, x^*)\right) \cdot M^*(x_n, x^*) \qquad (2.12)$$

$$M(x_n, x^*) = max\{d(x_n, x^*) + |d(x_n, Tx_n) - d(x^*, Tx^*)|,$$

$$d(x_n, x_{n+1}) + |d(x_n, x^*) - d(x^*, Tx^*)|,$$

$$d(x^*, Tx^*) + \left|d(x_n, x_{n+1}) - d\left(x_n, x^*\right)\right|,$$

$$\frac{1}{2}[d(x_n, Tx^*) + d(x^*, x_{n+1}) + |d(x_n, x_{n+1}) - d(x^*, Tx^*)|\} \qquad (2.13)$$

$$\lim_{n\to\infty} M(x_n, x^*) = d(x^*, Tx^*) \qquad (2.14)$$

Taking the limit as $n \to \infty$ in (2.12) and using (2.14) we get:

$$d(x^*, Tx^*) \leq \lim_{n\to\infty} \varphi\left(M(x_n, x^*)\right) \cdot d(x^*, Tx^*) < d(x^*, Tx^*)$$

$$\lim_{n\to\infty} \varphi\big(M(x_n,x^*)\big) = 1, \lim_{n\to\infty} M(x_n,x^*) = 0 \Rightarrow\ d(x^*,Tx^*) = 0$$

Hence, $x^* = Tx^*$.

Now, let us to show that T has at most one fixed point.

Indeed, if $x^*, y^* \in X$ are two distinct fixed points of T, that is, $Tx^* = x^* \neq y^* = Ty^*$, then

$$0 < d(x^*,y^*) = d(Tx^*,Ty^*) \leq \varphi\big(M(x^*,y^*)\big) \cdot M(x^*,y^*) \tag{2.15}$$

Because

$$M(x^*,y^*) = \max\{d(x^*,y^*) + |d(x^*,Tx^*) - d(y^*,Ty^*)|,$$

$$d(x^*,Tx^*) + |d(x^*,y^*) - d(y^*,Ty^*)|,$$

$$d(y^*,Ty^*) + |d(x^*,y^*) - d(x^*,Tx^*)|, \frac{d(x^*,Ty^*) + d(y^*,Tx^*) + |d(x^*,Tx^*) - d(y^*,Ty^*)|}{2}\} =$$

$= d(x^*,y^*)$, it follows from (2.15) that

$$0 < d(x^*,y^*) \leq \varphi\big(d(x^*,y^*)\big)d(x^*,y^*) < d(x^*,y^*).$$

This is a contradiction. Then $d(x^*,y^*) = 0$, so $x^* = y^*$. This proves than the fixed point of T is unique.

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